FINAL EXAM, ALGEBRA III, TOTAL MARKS: 50

(1) (10 marks) Let $R := \{a = (a_n)_{n\in\mathbb{N}} | a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ and there exists $n_0 \in \mathbb{N}$ (depending on a) such that $a_{n_0} = a_{n_0+k}$ for all $k \in \mathbb{N}\}$ be the set of sequences of real numbers which are eventually constant. R is a ring with addition and multiplication defined componentwise, that is, $a + b = (a_n + b_n)_{n\in\mathbb{N}}$ and $ab = (a_n b_n)_{n\in\mathbb{N}}$. Describe the set $\mathrm{Max}(R)$ of all maximal ideals of R.

(2) (15 = 8+7 marks) Let $\omega = e^{2i\pi/3} = \frac{-1+i\sqrt{3}}{2}$, let $R = \mathbb{Z}[\omega]$. Let p be a positive prime integer which is not equal to 3.
   (a) Prove that R is an Euclidean domain, and find all the units of R.
   (b) Prove that the ideal pR is a maximal ideal of R, if and only if, $p \equiv -1$ (modulo 3).

(3) (7 marks) Find a direct sum of cyclic groups isomorphic to the abelian group presented by the matrix

$$\begin{pmatrix} 4 & 12 & 12 \\ 8 & 4 & 16 \\ 16 & 16 & 8 \end{pmatrix}$$

Write the invariant factors and the elementary divisors of the group.

(4) (10 = 5 $\times$ 2 marks) Consider the real 3×3 matrix A

$$\begin{pmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ -3 & -3 & -5 \end{pmatrix}$$

   (a) Compute the characteristic polynomial of A.
   (b) Compute the minimal polynomial of A.
   (c) Find the rational canonical form of A.
   (d) Find the Jordan canonical form of A.
   (e) What are the invariant factors and elementary divisors of A?

(5) (8=5+3 marks) Prove that if A, B are two 3×3 matrices over a field F, then A and B are similar, if and only if, they have the same characteristic and minimal polynomial. Give an example to show that this is false for 4×4 matrices.